THE TWO-DIMENSIONAL PROBLEM OF FLUIDIZATION

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Iu.P. GUPALO and G.P. CHERENKOV
(Moscow)

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When a liquid, or gas flows through a particulate medium, its solid particles are subjected to hydrodynamic forces exerted by the stream, and to forces due to the interaction of adjacent particles. Under certain conditions of flow the medium disintegrates, its particles loose their regular contact with each other, and the medium (or a certain part of it) reaches the state of fluidization. It is of interest to determine the conditions of transition of the medium into that state.

Practical requirements for this are, in the first instance, due to the rapid development of the fluidized bed technique in the chemical industry. For reasons of technological effectiveness of processes, vessels of industrial apparatus are in the main given complicated shapes, with nonuniform fluid distribution through the vessel's cross section. Definite technological advantages are obtained with vessels of a cross section increasing towards the top [1 and 2]. In many processes (such as, for example, drying of particulate materials in a fluidized state) a gushing bed which is obtained, when a stream of gas is introduced from underneath into a packed bed, is used [3]. Finally, the problem of transition into the fluidized state of a sand slug in a borehole, a problem which is relevant to the forecasting of petroleum yield limit, may be quoted.

In the following we propose to formulate the problem of a particulate body transition into the fluidized state, as a problem of transient equilibrium of a body in which compressive stresses only can exist. Any strains present in it prior to the in stant of transition into the critical state are ignored, so that the body under consideration becomes in a sense analogous to a specific stiff-plastic body. A computation method has been evolved for the case of the two-dimensional problem, which permits a reasonably effective determination of the critical fluid flow parameters on which depends the transition of a medium into the fluidized state. The two-dimensional problem is in certain respects similar to the problem of local buckling of a membrane [4]. Solutions of several specific problems of transition of a medium into the fluidized state are given, and a comparison is made of theoretical solutions with published experimental data on fluidization. A quantitative and qualitative correlation between theory and experiment is revealed.

1. Statement of problem. Let us consider a particulate body in the interstices of which flows a liquid or gas. We shall use the concept of a continuous two-phase medium, i.e. we assume that at every point of space occupied by the particulate body and the fluid there are at the same time two continuous bodies, the state of one (the particulate body) being characterized by stresses and strains, and that of the other (the liquid or gas) by the pressure and velocity of moving particles. The interaction of these two media is determined experimentally, and is reduced to the introduction into each of these of certain volume forces which at every point of space must evidently be equal in magnitude, and of opposite direction. We assume that the particulate body voidage is fully determined by the fluid pressure, and the volume force by the fluid pressure and velocity. On these assumptions the fluid pressure and velocity may be determined independently of the stress and strain state of the particulate body. In the following analysis we shall consider this part of the problem, which belongs to the classic problems of the theory of filtration [5], as solved.

We shall analyze the stress and strain state of the particulate body under the action of volume forces known from the solution of the corresponding filtration theory problem. We shall confine our analysis to a two-dimensional problem. The Eqs. of a particulate body static equilibrium are

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = a, \qquad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial y} = b \tag{1.1}$$

Here, σ_x, σ_y, τ_{xy} are components of the stress tensor in an orthogonal system of Carte-coordinates xy, a (x, y) and b (x, y) are components of the volume force vector along the x- and y-axes respectively, taken with the opposite sign. We emphasize that stresses σ_x, σ_y, τ_{xy} characterize the forces of the particulate body particles interaction. We apply the fundamental concept of the transient equilibrium theory, according to which at every point of a continuous medium in a critical state is fulfilled a certain relationship of the form

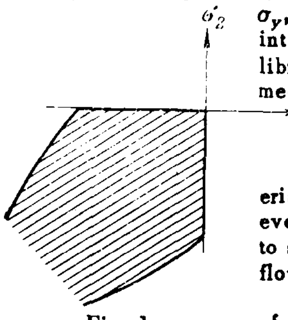

Fig. 1

$$F(\sigma_x, \sigma_y, \tau_{xy}) = 0 \tag{1.2}$$

Here, function F depends on stresses and constants of the material. It is natural to make use of the von Mises postulate; then at every point of the body the strain increment vector will be normal to surface (1.2) at the corresponding point (associative law of flow). We disregard the body elastic strains.

We shall now consider the selection of function F for the case of a particulate body. Let the two main stresses be compressive at every point of a certain area of the body. The Coulomb's law is generally applied in such cases [6]

$$\tfrac{1}{4}(\sigma_x - \sigma_y)^2 + \tau_{xy}^2 = \tfrac{1}{4}\sin^2\delta\,(\sigma_x + \sigma_y + K \operatorname{ctg}\delta)^2 \tag{1.3}$$

Here δ and K are respectively, the angle ot internal friction, and the coefficient of adhesion.

We note that the medium considered here consists of separate particles, only loosely bound to each other. In the first approximation the forces of particles adhesion, which arise when the body is in tension, may be neglected. With this assumption the medium continuity will be disturbed by an application of any arbitrarily small tension force, i.e. the body will disintegrate. It follows from these considerations that function F expressed in terms of the main stresses σ_1 and σ_2 will be of the form shown on Fig. 1, where the medium in the shaded area is considered to be undeformable, and the state outside that area unattainable. The only problems which were considered in the theory of transient equilibrium of particulate bodies, were those in which the two main stresses were compressive [6].

We shall analyze problems in which only the states along axes σ_1 and σ_2 (Fig. 1) are possible, i.e. on condition

$$\tau_{nt} \leqslant K + \sigma_n \operatorname{tg}\delta \tag{1.4}$$

where τ_{nt} and σ_n are respectively the tangent and normal stresses on any elementary plane within the packed layer.

The latter occurs, for example, in the majority of technological problems concerning the transient equilibrium of a packed bed in a stream of liquid or gas. We note that the model of a medium which does not resist tension stresses was first proposed in [4] in connection with some other problems.

The main stresses in a two-dimensional problem are:

$$\left.\begin{matrix}\sigma_{\max}\\ \sigma_{\min}\end{matrix}\right\} = \tfrac{1}{2}(\sigma_x + \sigma_y) \pm \tfrac{1}{2}\sqrt{(\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2} \tag{1.5}$$

According to Fig. 1 the greatest of the main stresses under conditions of transient equilibrium is equal zero. Hence,

$$\sigma_x \sigma_y = \tau_{xy}^2 \tag{1.6}$$

The stated problem is reduced to the analysis of a quasi-linear system of Eqs. (1.1), (1.6). Condition (1.6) may be formally derived from Coulomb's law with $\delta = \tfrac{1}{2}\pi$. However, the mathematical construction changes considerably in this limiting case due to the change of the type of system (of [6] and Sections 2 and 3 of this paper).

2. System analysis. 1°. Let the stresses be of the form

$$\sigma_x = -\alpha^2(x, y), \quad \sigma_y = -\beta^2(x, y), \quad \tau_{xy} = -\alpha(x, y)\beta(x, y) \tag{2.1}$$

Eq. (1.6) is fulfilled, and system (1.1) becomes

$$2\alpha\frac{\partial\alpha}{\partial x} + \alpha\frac{\partial\beta}{\partial y} + \beta\frac{\partial\alpha}{\partial y} = -a, \quad \alpha\frac{\partial\beta}{\partial x} + \beta\frac{\partial\alpha}{\partial x} + 2\beta\frac{\partial\beta}{\partial y} = -b \tag{2.2}$$

We shall solve the Cauchy problem for system (2.2). Let arc $x = x_0(s)$, $y = y_0(s)$, be specified in the xy-plane, for which functions $\alpha = \alpha_0(s)$, $\beta = \beta_0(s)$ are known. Differentiating the two latter functions with respect to the arc length s, we obtain two conditions for the 'strip'

$$\frac{\partial\alpha}{\partial x}dx + \frac{\partial\alpha}{\partial y}dy = d\alpha, \qquad \frac{\partial\beta}{\partial x}dx + \frac{\partial\beta}{\partial y}dy = d\beta \tag{2.3}$$

Eqs. (2.2) and (2.3) represent a system of four linear algebraic equations with respect to the four unknown derivatives.

Equating the determinant of this system to zero, we find

$$\alpha\, dy = \beta\, dx \tag{2.4}$$

Thus, system (2.2) has one set of characteristics, and is of the parabolic type. We shall find the relationship which must be fulfilled along the characteristic. With this in view, we specify the characteristic as a line of weak discontinuity. In this case the numerators in Cramer's formulas must vanish, and we obtain the relationship

$$\alpha^2\frac{d^2y}{dx^2} = a\frac{dy}{dx} - b \tag{2.5}$$

We shall establish the physical meaning of the characteristics. By virtue of (1.5) and (2.1) we have at every point of the body

$$\sigma_{\max} = 0, \qquad \sigma_{\min} = \sigma_x + \sigma_y = -(\alpha^2 + \beta^2)$$

We shall calculate angle θ between the area element of the maximum main stress, which is equal zero, and the x-axis. We have

$$-\sigma_x \sin\theta + \tau_{xy}\cos\theta = 0$$

From this, using Formulas (2.1) and (2.4), we obtain

$$\operatorname{tg}\theta = dy / dx \tag{2.6}$$

The characteristic is, therefore, a line along which normal and tangential stresses are zero.

We shall take advantage of this property when analyzing condition (1.4). Let σ_ξ be a stress along characteristics. Then, for any area element with its normal at angle γ to the tangent of the characteristic at a given point, we have

$$\tau_{nt} = {}^1\!/_2 \sin 2\gamma\,(-\sigma_\xi), \quad \sigma_n = \cos^2\gamma\,(-\sigma_\xi) \tag{2.7}$$

It follows from (2.7) that condition (1.4) will be fulfilled for any γ, when

$$\delta \geqslant 2\operatorname{arctg}\frac{1 - 2K/|\sigma_\xi|}{1 + 2K/|\sigma_\xi|} \tag{2.8}$$

2°. In any arbitrary orthogonal curvilinear coordinates ξ and η the initial system (1.1) will be of the form [7]

$$H_1Q_\xi + \frac{1}{H_1H_2}\frac{\partial}{\partial\xi}(H_1H_2\sigma_\xi) + \frac{1}{H_1H_2}\frac{\partial}{\partial\eta}(H_1^2\tau_{\xi\eta}) - \sigma_\xi\frac{\partial\ln H_1}{\partial\xi} - \tau_{\xi\eta}\frac{\partial\ln H_2}{\partial\xi} = 0 \tag{2.9}$$

$$H_2Q_\eta + \frac{1}{H_1H_2}\frac{\partial}{\partial\xi}(H_2^2\tau_{\xi\eta}) + \frac{1}{H_1H_2}\frac{\partial}{\partial\eta}(H_1H_2\sigma_\eta) - \tau_{\xi\eta}\frac{\partial\ln H_1}{\partial\eta} - \sigma_\eta\frac{\partial\ln H_2}{\partial\eta} = 0$$

Here, σ_ξ, σ_η, $\tau_{\xi\eta}$ are stress components in the $\xi\eta$-coordinate system, Q_ξ and Q_η are components of the volume force vector along axes ξ and η, and H_1 and H_2 the Lamé parameters

$$H_1^2 = (dx / d\xi)^2 + (\partial y / \partial\xi)^2, \qquad H_2^2 = (\partial x / \partial\eta)^2 + (\partial y / \partial\eta)^2 \tag{2.10}$$

The unknown equations of the set of characteristics are denoted by $x = x(\xi, \eta)$, $y = y(\xi, \eta)$ (condition $\eta = \text{const}$ isolates one characteristic of the set).

Now let lines $\eta = \text{const}$ be characteristics of the initial system (1.1), (1.6). As along characteristics $\sigma_\eta = 0$, $\tau_{\xi\eta} = 0$, Eqs. (2.9) may be written in the following form, equivalent to the system of Eqs. (1.1), (1.6)

$$H_1 Q_\xi + \frac{\partial \sigma_\xi}{\partial \xi} + \sigma_\xi \frac{\partial \ln H_2}{\partial \xi} = 0, \qquad Q_\eta = 0 \tag{2.11}$$

Components Q_ξ and Q_η can obviously be expressed in terms of a and b

$$Q_\xi = -a \frac{\partial x}{\partial \xi} - b \frac{\partial y}{\partial \xi}, \qquad Q_\eta = a \frac{\partial y}{\partial \xi} - b \frac{\partial x}{\partial \xi} \tag{2.12}$$

For ξ we simply take the length of arc of characteristic $\eta = \text{const}$. Then $H_1{}^2 = 1$, or

$$(\partial x / \partial \xi)^2 + (\partial y / \partial \xi)^2 = 1 \qquad \text{for } \eta = \text{const} \tag{2.13}$$

By virtue of (2.12) the second of Eqs. (2.11) may be written

$$a \partial y / \partial \xi = b \partial x / \partial \xi \tag{2.14}$$

Taking into account (2.13) and (2.14) we find that along characteristic $\eta = \text{const}$

$$\left(\frac{\partial x}{\partial \xi}\right)^2 = \frac{a^2}{a^2 + b^2}, \qquad \left(\frac{\partial y}{\partial \xi}\right)^2 = \frac{b^2}{a^2 + b^2} \tag{2.15}$$

Stipulating for the set of characteristics an equation of the form $y = y(x, \eta)$, and using (2.14), we readily derive the simple Eq.

$$dy / dx = b / a \qquad \text{for } \eta = \text{const} \tag{2.16}$$

Eq. (2.16) is used for the determination of the unknown set of characteristics (the arbitrary constant appearing in the solution of Eq. (2.16) is denoted above by η).

We rewrite the first of Eqs. (2.11) in the form

$$H_2 Q_\xi + \frac{\partial}{\partial \xi} (H_2 \sigma_\xi) = 0 \tag{2.17}$$

Bearing in mind Formula (2.12), we integrate (2.17)

$$H_2 \sigma_\xi = \int_{\xi_0}^{\xi} H_2(\xi) \left(a \frac{\partial x}{\partial \xi} + b \frac{\partial y}{\partial \xi}\right) d\xi + A(\eta) \tag{2.18}$$

Here the function $A(\eta)$ is determined from the boundary conditions.

If the set of characteristics is given in the form of $x = x(\xi, \eta)$, $y = y(\xi, \eta)$, then by virtue of (2.10), (2.15) the only unknown stress σ_ξ is finally derived in the following form:

$$\sigma_\xi = \frac{1}{\sqrt{R(\xi, \eta)}} \left(A(\eta) + \int_{\xi_0}^{\xi} \sqrt{(a^2 + b^2) R(\xi, \eta)}\, d\xi\right), \qquad R(\xi, \eta) = \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 \tag{2.19}$$

The selection of the radical sign, here and throughout the following analysis, is best made on the ground of physical considerations.

We shall now turn to a more convenient presentation of the set of characteristics in the form of $y = y(x, \eta)$. Having written down the result of differentiation of the identity $y \equiv y(x(\xi, \eta), \eta)$ with respect to η, together with the orthogonality condition of coordinates $\xi\eta$, we obtain system

$$\frac{dy}{dx} \frac{\partial x}{\partial \eta} + \frac{dy}{d\eta} = \frac{\partial y}{\partial \eta}, \qquad \frac{\partial x}{\partial \xi} \frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi} \frac{\partial y}{\partial \eta} = 0$$

Here, dy/dx and $dy/d\eta$ denote, as usual, partial derivatives of function $y = y(x, \eta)$. This notation has already been used above in Eq. (2.16). Solving this system for $\partial x/\partial \eta$ and $\partial y/\partial \eta$ and using Formulas (2.15) and (2.16), we obtain

$$\frac{\partial x}{\partial \eta} = -\frac{ab}{a^2 + b^2} \frac{dy}{d\eta}, \qquad \frac{\partial y}{\partial \eta} = \frac{a^2}{a^2 + b^2} \frac{dy}{d\eta}$$

Thus, when the set of characteristics is specified in the form of $y = y(x, \eta)$, then on the basis of (2.19) the σ_ξ-stress may be written down in one of the following simple forms

$$\sigma_\xi = \frac{\sqrt{a^2 + b^2}}{a} \left(\frac{dy}{d\eta}\right)^{-1} \left[A(\eta) + \int_{x_0}^{x} \sqrt{a^2 + b^2} \frac{dy}{d\eta} dx\right] \tag{2.20}$$

or

$$\sigma_\xi = \frac{\sqrt{a^2 + b^2}}{a} \left(\frac{dy}{d\eta}\right)^{-1} \left[A(\eta) + \int_{y_0}^{y} \frac{a}{b} \sqrt{a^2 + b^2} \frac{dy}{d\eta} dy\right] \tag{2.21}$$

3. The boundary value problem. 1°. The most convenient computation procedure for solving specific problems according to the foregoing is this: first of all, the field of characteristics is analyzed with the aid of Eq. (2.16), then, using one of the Formulas (2.19), (2.20) or (2.21), the body stress field is determined so as to satisfy the boundary conditions along the perimeter of the body.

It is interesting to compare the results derived from the analysis of system (1.1), (1.6) with those obtained earlier in [4] for the same system, but with $a = b = 0$.

The system of Eqs. (1.1), (1.6) is quasi-linear when $a \neq 0$, $b \neq 0$, and belongs to the parabolic type, its only set of characteristics is fully defined by coefficients $a(x, y)$ and $b(x, y)$, and therefore is explicitly independent of the body configuration and of the boundary conditions. In the degenerate case, when $a = b = 0$, the system is also quasi-linear, and belongs to the parabolic type, but its set of characteristics is fully determined by the form of the body, and by the boundary conditions.

In view of these peculiarities of the sets of characteristics, it is necessary to consider the question of the statement of a correct boundary value problem for these equations, i.e. a problem which would guarantee the existence and uniqueness of its solution, as well as in a certain sense its stability with respect to perturbations of the body configuration and of the boundary conditions. It will be readily seen that the only correctly stated problem for system (1.1), (1.6) with $a \neq 0$, $b \neq 0$, will be the boundary value problem for an arbitrary body boundary with surface forces at an angle arc tg(b/a) to the x-axis (i.e. surface loads must be directed along the characteristics). In fact, a solution cannot exist, if the loads at the body boundary were orientated in any other way, because the equilibrium conditions (1.1) cannot be satisfied for medium (1.6) even in the neighborhood of the boundary. On the other hand, if the stipulated condition as to the direction (of loads) is maintained, the boundary value problem will be correct for any single-valued functions a and b, and for any arbitrary configuration of the body. This follows from the analysis of the equation of characteristics (2.16) and Formula (2.20) for stress $\sigma_\xi = \sigma_x + \sigma_y$, taking into account that σ_ξ can be determined along each of the characteristics independently of the field of σ_ξ in the remaining area.

We remember that in the degenerate case with $a = b = 0$ any Cauchy problem will be a correct boundary value problem (provided that the characteristics do not intersect) [4]. In connection with the consideration of correct boundary value problems for systems (1.1), (1.2) of the transient equilibrium theory, it should be pointed out, in so far as this had not been previously mentioned in the literature, that with the transient equilibrium conditions (1.2) presented in the form

$$F(p, \tau_m) = 0, \; p = {}^1/_2(\sigma_x + \sigma_y), \; \tau_m = \sqrt{{}^1/_4(\sigma_x - \sigma_y)^2 + \tau_{xy}{}^2} \qquad \left(\frac{\partial F / \partial p}{\partial F / \partial \tau_m} > 1\right) \tag{3.1}$$

which lead to systems of an elliptical type [8], the Cauchy boundary value problem is not correct. In particular, this case occurs in certain problems of the theory of two-dimensional plastic state of stress [9 and 10].

From this point of view Hill's concept of the possibility of existence of plastic states corresponding to circles entirely contained within Mohr's envelope does not appeal to be true ([8] p. 337).

2°. We shall adduce an example of correctly stated boundary value problem for system (1.1), (1.6). Let there be, inside a heavy particulate medium contained in space D, a vessel filled with gas, or liquid occupying space D_+ (Fig. 2). Boundary C_1 represents a rigid perforated screen impervious to solid particles. With increasing pressure in vessel D_+ the gas begins to filter through the particulate body into space D_- containing gas at a lower pressure At a certain pressure drop the medium reaches a critical state, after which it disintegrates, and the bed begins to 'boil'.

At pressure differentials smaller than, or equal to the critical the internal stresses of the medium are determined by Formulas (2.20) and (2.16). A change of the pressure drop is generally accompanied by a change of the field of characteristics. We separate an arbitrary elementary strip AB of the particulate body situated between adjacent characteristics (Fig. 2). With very small pressure differentials between points A and B the σ_ξ-stress along the whole length of the strip is compressive, except at point B, where σ_ξ is zero at any pressure drop. The latter is due to the σ_ξ-stress being a characteristic of the interaction of particles of the particulate body (see Note to (1.1)), which becomes evident if an arbitrarily small tensile stress is added to any hydrostatic pressure at point B of the particulate body surface. It is clear from physical considerations that with an increase of the pressure drop at a certain point E of the particulate body the σ_ξ-stress will vanish. At that instant the volume forces acting on the elementary strip BE in the proximity of characteristic AB passing through point E will obviously be in equilibrium.

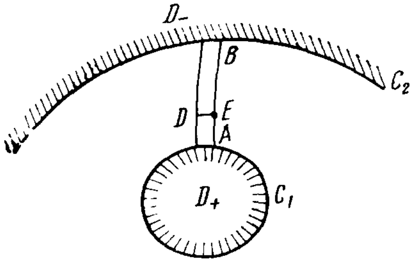

Fig. 2

A further increase of the pressure drop will generally result in an upward dynamic motion of the elementary column BE. This instant may well be considered as the beginning of the disintegration of the packed bed, and its transition into a fluidized state.

Thus, the unknown parameters of the critical state of a particulate body in a liquid, or gas are determined by the condition that the σ_ξ-stress must vanish at a certain point within the body, with compressive σ_ξ-stresses at all other points. This condition, although necessary, is generally not sufficient for the onset of disintegration (a case of indefinite equilibrium is possible). The case of gradual formation of an area of zero stresses in the surface neighborhood requires further study.

It will be easy to deduce from the above example the kinematic pattern of the velocity field distribution with the use of the associative law of flow, and disregarding elastic strains. From the associative law we obtain equalities $\varepsilon_\xi{}' = 0, \gamma_{\xi\eta}{}' = 0$, where $\varepsilon_\xi{}'$, and $\gamma_{\xi\eta}{}'$ are the corresponding strain rate components. For these components we obtain a homogenous linear system of two partial differential equations of the first order, for which it will be necessary to solve the Cauchy problem with respect to specified velocities at screen C_1. Hence, if this screen is stationary, the particle velocities in the precritical stage will be zero throughout the body. At transition through the critical state the equilibrium of the elementary column BE will be violated.

3°. It should be noted that the stated critical equilibrium condition is also valid in the presence of solid walls, the introduction of which is, however, subject to two conditions: firstly, the wall must coincide with a streamline (filtration problem requirement), and secondly, the following inequality must be satisfied along the wall

$$\tau_{nt}{}^{(w)} \leqslant K_w + \sigma_n{}^{(w)} \operatorname{tg} \delta_w \tag{3.2}$$

Here, $\tau_{nt}{}^{(w)}, \sigma_n{}^{(w)}$ are respectively the tangent and the normal stresses at the wall, δ_w is the angle of friction of the system particulate material – wall, and K_w the adhesion coefficient of the particulate material to the wall.

Condition (3.2) imposes certain limitations as regards angle γ_w between the tangent to the characteristic at the intersection point of the latter with the wall, and the normal to the wall at that point. In fact, we have for $\tau_{nt}{}^{(w)}$ and $\sigma_n{}^{(w)}$

$$\tau_{nt}{}^{(w)} = {}^1/_2 \sin 2\gamma_w (-\sigma_\xi), \qquad \sigma_n{}^{(w)} = \cos^2 \gamma_w (-\sigma_\xi) \tag{3.3}$$

Substituting (3.3) into (3.2) we obtain condition

$$\frac{\cos \gamma_w \sin (\gamma_w - \delta_w)}{\cos \delta_w} \leqslant \frac{K_w}{-\sigma_\xi} \tag{3.4}$$

Condition (3.4) is fulfilled for any K_w, δ_w, and σ_ξ, when $\gamma_w = \frac{1}{2}\pi$, i.e. when a characteristic coincides with the wall. In the absence of such coincidence, (3.4) is valid for any K_w and σ_ξ when $\gamma_w \leqslant \delta_w$, or for any γ_w and

$$\delta_w \geqslant 2 \operatorname{arctg} \frac{1 - 2K_w / |\sigma_\xi|}{1 + 2K_w / |\sigma_\xi|} \tag{3.5}$$

4. Certain one-dimensional problems of transient equilibrium. 1°. Turning to the solution of specific problems of transient equilibrium of beds in a stream of fluid, we shall first of all note that the resultant volume force acting on the particulate body, the components of which appear with opposite signs in the right hand sides of Eqs. of static equilibrium (1.1), is

$$\mathbf{Q} = \rho^* \mathbf{F}^* - \operatorname{grad} p \qquad (a = -Q_x, \; b = -Q_y) \tag{4.1}$$

Here, ρ^* is the density of the two-phase system of packed bed and fluid, and $\mathbf{Q}$ the external mass force acting on the particulate body saturated by the fluid, and

$$\rho^* = \varepsilon\rho + (1-\varepsilon)\rho', \qquad \rho^*\mathbf{F}^* = \varepsilon\rho\mathbf{F} + (1-\varepsilon)\rho'\mathbf{F}' \tag{4.2}$$

Here, ε is the voidage of the particulate body, ρ and ρ' are the respective densities of fluid and of solid particles, and $\mathbf{F}$ and $\mathbf{F}'$ are vectors of external mass forces acting on the fluid and the solid phase respectively. Generally $\mathbf{F} \neq \mathbf{F}'$ (for example, under the influence of a magnetic field). However, in practice usually $\mathbf{F} \equiv \mathbf{F}'$, and in that case we have from (4.2) $\mathbf{F}^* \equiv \mathbf{F}$.

The pressure gradient in (4.1) is found by solving the filtration problem, i.e. by solving with appropriate boundary conditions [5] the following system

$$\operatorname{grad} p = -\frac{\mu}{k}\mathbf{u}\Phi(|u|) + \rho\mathbf{F}, \qquad \operatorname{div}\mathbf{u} = 0 \tag{4.3}$$

Here μ is the fluid viscosity, k is the permeability coefficient, and $\Phi(|u|)$ a function of the filtration rate module $\mathbf{u}$ which determines the stipulated (generally nonlinear) filtration law. In practice the following linear approximation is generally used:

$$\Phi(|u|) = 1 + \lambda |u| \tag{4.4}$$

The dependence of the permeability coefficient k, and of coefficient λ appearing in (4.4) on the fluid properties, on the dimensions of the bed particles, and on the bed voidage is established by means of dimensional analysis with an approximation of the order of the numerical factor. In fact,

$$|\operatorname{grad} p| \sim \mu^{1-n} l^{n-2} \rho^n u_\varepsilon^{1+n}$$

where l is a characteristic interstice dimension, and u_ε the intersticial fluid velocity. If following Kozeny, we define l in the same manner as the hydrodynamic diameter of a channel, then $l = {}^1/_6\, D\varepsilon(1-\varepsilon)^{-1}$, where D is the equivalent diameter of a particle of the bed, equal to the diameter of an isometric sphere. From the continuity condition we have $u_\varepsilon = u/\varepsilon$. Hence,

$$|\operatorname{grad} p| \sim \mu^{1-n} D^{n-2} \rho^n \varepsilon^{-1-n} (1-\varepsilon)^{2-n} u^{1+n}$$

If $uD\rho\mu^{-1} \ll 1$, then grad p should be independent of ρ, therefore $n = 0$. On the other hand, when $uD\rho\mu^{-1} \gg 1$, then grad p should be independent of μ, i.e. in this case $n = 1$. We may therefore conclude that

$$\frac{\mu}{k} = c_1 \frac{\mu}{D^2}\frac{(1-\varepsilon)^2}{\varepsilon^3}, \qquad \lambda = c_2 \frac{\rho D}{\mu}\frac{1}{1-\varepsilon} \tag{4.5}$$

where c_1 and c_2 are numerical constants which may be determined by using, for example, the results obtained by Ergun [11], who had processed a considerable amount of experimental data collected by other authors. According to [11], these constants are: $c_1 = 150$, $c_2 = {}^1/_{87.5}$.

2°. The pattern of transition of a bed into the fluidized state is simplest, when the characteristics of system (1.1), as defined by (2.16), coincide with streamlines. We shall first consider the case in which the bed is in a gravitational field, and is contained between vertical flat walls, with the fluid uniformly fed from below through the bed cross section. We direct the x-axis of an orthogonal Cartesian coordinate system along a normal to the wall, and the y-axis in the direction opposite to the gravity force, and assume that the charge occupies a space defined by $0 \leqslant y \leqslant H$ (plane $y = 0$ is permeable to the fluid, but impervious to particles).

The solution of system (4.3) is: $u \equiv 0$, $v =$ const, and $\partial p/\partial x = 0$. According to (2.16) the characteristics in this case are straight lines $x =$ const, hence, $\sigma_x = \tau_{xy} = 0$. Using the condition that at the free surface $y = H$, $\sigma_y = 0$, we derive from Formulas (1.1) and (4.1)

$$\sigma_y = -\int_y^H (\partial p/\partial y + \rho^* g)\, dy \tag{4.6}$$

If the parameters of a bed, i.e. its voidage, density, and the size of its particles do not vary along its height, then obviously ρ^* and $\partial p/\partial y =$ const. It follows then from (4.6) that the transition of the bed into a fluidized state takes place simultaneously throughout its volume, and that the condition for the occurance of transition is defined by

$$-\partial p/\partial y = \rho^* g \tag{4.7}$$

Condition (4.7) in the case of uniformly packed beds is well supported by experiment. The expression for the minimum filtration rate may be readily derived from (4.3) and (4.7).

The case of a uniformly packed bed was considered above. However, in practice cases of beds not packed uniformly along the column height are not infrequent. Such inhomogenuity is usually connected with a nonuniform distribution of voidage along the height, and is due to local compaction of the bed. This lack of uniformity considerably affects the conditions and character of the bed transition into the fluidized state. Let there be, for example, in a uniformly packed bed with voidage ε_0 an area of local compaction defined by $h \leqslant y \leqslant h + \Delta h$ ($h > 0$, $h + \Delta h < H$) with voidage $\varepsilon_1 < \varepsilon_0$. We shall analyze the equilibrium condition for this bed. From (4.6) and (4.5) it follows that

$$\sigma_y = \begin{cases} \sigma_y{}^0(y), & h + \Delta h < y \leqslant H \\ \sigma_y{}^0(h+\Delta h) + \sigma_y{}^1(y), & h \leqslant y \leqslant h + \Delta h \\ \sigma_y{}^0(y+\Delta h) + \sigma_y{}^1(h), & 0 \leqslant y < h \end{cases} \tag{4.8}$$

where, by virtue of (4.4), we have

$$\sigma_y{}^0(y) = [(\mu / k_0)\, v\, (1 + \lambda_0 v) - (\rho' - \rho)(1 - \varepsilon_0)\, g]\,(H - y)$$
$$\sigma_y{}^1(y) = [(\mu / k_1)\, v\, (1 + \lambda_1 v) - (\rho' - \rho)(1 - \varepsilon_1)\, g]\,(h + \Delta h - y) \tag{4.9}$$
$$(k_i = k(\varepsilon_i),\ \lambda_i = \lambda(\varepsilon_i),\ i = 0, 1)$$

Substitution of functions $k(\varepsilon_i)$ and $\lambda(\varepsilon_i)$ defined by (4.5) into (4.8) shows that the critical state of the nonuniform bed considered, which corresponds to σ_y becoming zero, occurs first of all at $y = h$, i.e. at the lower boundary of the compacted layer.

The following Eq. of the critical velocity v_* corresponds to this condition:

$$\left(\frac{H-h-\Delta h}{k_0}\lambda_0 + \frac{\Delta h}{k_1}\lambda_1\right) v_*^2 + \left(\frac{H-h-\Delta h}{k_0} + \frac{\Delta h}{k_1}\right) v_* = \left[(H-h)(1-\varepsilon_0) + \Delta h\,(\varepsilon_0 - \varepsilon_1)\right] \frac{\rho' - \rho}{\mu} g \tag{4.10}$$

It follows specifically from (4.10) that $v_{*1} < v_* < v_{*0}$ where v_{*0} and v_{*1} are the critical filtration rates corresponding to the transient equilibrium of uniformly packed layers with voidages ε_0 and ε_1. The distribution of stress $\sigma_y(y)$ along the height of the nonuniform charge (4.8) and (4.9) has been plotted on Fig. 3, where curves *1*, *2*, *3* and *4* relate to filtration velocities $v < v_{*1}$, $v = v_{*1}$, $v_{*1} < v < v_*$, $v = v_*$.

The condition of the nonuniformly packed bed critical state is fulfilled only locally, hence, discontinuities along the lower boundary of local compaction must be expected in such beds. The formation of these discontinuities in the form of distinctive cracks had been observed experimentally [12].

It seems that the formation of local compacted layers of a bed followed by ruptures, is the basic mechanism of the slug formation process which can be observed during fluidization of beds of considerable height in tubes of small diameters.

3°. In Subsection 2° the mass force F was assumed to be constant throughout the bed volume. This condition however may not always be satisfied. As an example, we may quote the case of a rotating fluidized bed (the axis of rotation being normal to the plane of flow) used in rocketry [13], and in chemical technology [14]. In this case the bed has the form of a ring lying on the inner surface of a circular cylinder rotating around its axis, with fluid being uniformly fed through the porous cylinder wall impervious to solid particles, and flowing radially towards the axis of rotation, i.e. the field of flow in a two-dimensional problem is defined by a sink at the axis of rotation. By superposing the axis of rotation on the coordinate origin we obtain from (4.1) and (4.3) the following expressions for a and b:

$$a = \frac{\mu}{k}\frac{q}{2\pi}\frac{x}{x^2+y^2}\,\Phi\left(\frac{q}{2\pi}\frac{1}{\sqrt{x^2+y^2}}\right) - (1-\varepsilon)(\rho'-\rho)\,\omega^2 x$$
$$b = \frac{\mu}{k}\frac{q}{2\pi}\frac{y}{x^2+y^2}\,\Phi\left(\frac{q}{2\pi}\frac{1}{\sqrt{x^2+y^2}}\right) - (1-\varepsilon)(\rho'-\rho)\,\omega^2 y \tag{4.11}$$

where ω is the angular velocity of rotation, and q the flow rate.

From (2.16) and (4.11) it follows that the characteristics are straight lines passing through the coordinate origin, i.e. the characteristics coincide with streamlines. Introducing a system of polar coordinates r, $r\varphi$, we obtain from (2.19)

$$\sigma_r = \frac{1}{r}\int_{r_0}^{r} \left\{\frac{\mu}{k}\frac{q}{2\pi}\,\Phi\left(\frac{q}{2\pi r}\right) - (1-\varepsilon)(\rho'-\rho)\,\omega^2 r^2\right\} dr \tag{4.12}$$

As $\Phi(|u|)$ is usually a monotonously increasing function, the change of the σ_r-stress sign due to an increasing flow rate q will first occur, in accordance with (4.12), at the boundary $r = r_0$. The corresponding minimum flow rate q_* is obviously defined by the condition $(\partial\sigma_r/\partial r)_{r=r_0} = 0$. We obtain

$$\frac{\mu}{k}\frac{q_*}{2\pi r_0}\Phi\left(\frac{q_*}{2\pi r_0}\right) = (1-\varepsilon)(\rho' - \rho)\,\omega^2 r_0 \quad (4.13)$$

Transition to the fluidized state is gradual, starting at the surface of the bed. This is consistent with physical concepts, and is confirmed experimentally [14].

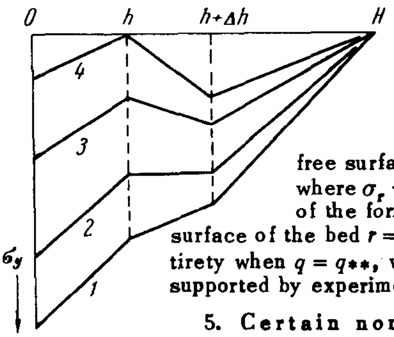

Fig. 3

We note that increasing the flow rate beyond $q_* = q_*(r_0)$ results in a gradual 'erosion' of the inner surface of a stationary bed, while the field of flow and that of characteristics remain unchanged. The free surface which separates the stationary layer of the bed, where $\sigma_r < 0$, from the fluidized one where $\sigma_r = 0$, is as before of the form $r =$ const, and gradually approaches the external surface of the bed $r = R$. In this case the bed becomes fluidized in its entirety when $q = q_{**}$, where $q_{**} = q_*(R)$. Values of q_* and q_{**} are well supported by experimental data [14].

5. **Certain non-one-dimensional problems.** 1°. Let us consider the two-dimensional problem of transient equilibrium of a bed in the gravitational field contained in vessel of a cross section increasing towards its top.

Let the packed bed be comprised between two intersecting planes under angle $\varkappa$ to the vertical, bounded at the bottom by a cylindrical surface $x^2 + y^2 = r_c^2$ permeable to the fluid, but impervious to particles, with the upper free surface also of a cylindrical form $x^2 + y^2 = R^2$. The fluid is uniformly fed through the permeable surface from underneath. In this case the fluid flow field is defined by its source situated at the coordinate origin. For simplicity's sake we shall limit our considerations to the linear law of filtration, and derive from (4.1) and (4.3) the following expressions for a and b:

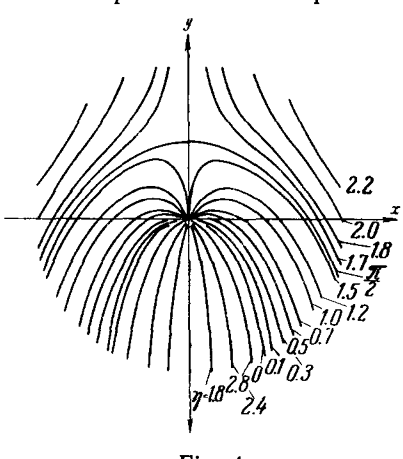

Fig. 4

$$a = -\frac{\mu}{k}\frac{q}{\varkappa}\frac{x}{x^2+y^2}$$

$$b = -\frac{\mu}{k}\frac{q}{\varkappa}\frac{y}{x^2+y^2} + (1-\varepsilon)(\rho'-\rho)\,g \quad (5.1)$$

where q is the fluid flow rate. Integrating the Eq. of characteristics (2.16), we obtain the following one-parameter set

$$y = x\,\mathrm{tg}\,(-Gx + \eta)$$

$$G = (1-\varepsilon)(\rho'-\rho)\,g\left(\frac{\mu}{k}\frac{q}{\varkappa}\right)^{-1} \quad (5.2)$$

which is represented on Fig.4 by solid lines. The field of characteristics has two singular points: a subcritical node at the coordinate origin (0, 0) and a saddle at point (0, $1/G$).

Substituting (5.1) and (5.2) into (2.21), we find that for the characteristics which emerge through the free surface the expression of the σ_ξ-stress is

$$\sigma_\xi = \frac{\mu q}{k\varkappa}\left(\frac{1-2Gy}{x^2+y^2} + G^2\right)^{1/2}\int\limits_{y_0}^{y}\frac{\sqrt{1-2Gy+G^2(x^2+y^2)}}{G(x^2+y^2)-y}\sqrt{x^2+y^2}\,dy \quad (5.3)$$

where $y_0 = \sqrt{R^2 - x_0^2}$ is the equation of the free surface. The integral in (5.3) is expressed by elliptic functions.

From physical considerations it follows that the critical state is first reached on the line of symmetry $x = 0$. Passing in (5.3) to the limit $x \to 0$, we obtain

$$\sigma_\xi = \left[\frac{\mu q}{k\varkappa}\frac{1}{y} - (1-\varepsilon)(\rho'-\rho)\,g\right](R-y) \quad \text{for } x = 0 \quad (5.4)$$

It follows from 5.4) that the minimum flow rate q_*, sufficient for σ_ξ to vanish inside the bed, corresponds to $y = r_0$, and is

$$q_* = \frac{\varkappa k}{\mu}(1-\varepsilon)(\rho'-\rho)gr_0 \tag{5.5}$$

It will be seen from (5.5) that in the subcritical state, i.e. when $q < q_*$, the singular point (0, $1/G$) lies below the lower surface of the bed, while in the critical state, i.e. when $q = q_*$ it coincides with the central point of that surface.

On the basis of the above considerations we obtain, in this case, the following pattern of transition of a bed into the fluidized state. At small rates of flow the field of characteristics is of the form shown on Fig. 4, with the saddle point below the lower surface. Along the characteristics inside the bed the σ_ξ-stresses are negative. An increasing flow rate is accompanied by a continuously changing field of characteristics, with the saddle point moving upwards along the axis of symmetry, until it reaches the lower surface of the bed at $q = q_*$. At that instant the σ_ξ-stress vanishes at point $(0, r_0)$ of the bed, and column $x = 0$, $r_0 \leqslant y \leqslant R$ begins to move upwards. The field of flow, and consequently the field of characteristics undergo a change. A further increase of the flow rate results in an upward movement of particles along the axis of the vessel, which is compensated by a downward motion of particles along the wall, i.e. a circulation of the solid phase is initiated which is so characteristic of fluidized beds in vessels of cross sections increasing towards the top [2 and 15], as well as of spouting beds [3 and 15].

2°. It is not difficult to examine the problem of transient equilibrium of a packed bed in a vessel tapering upwards by following the procedure of 1°. The difference between this problem, and that considered in Subsection 1° is as follows: the direction of the force of gravity has been reversed; the top free surface and the lower surface permeable to fluid only, are now expressed by Eqs. $x^2 + y^2 = r_0^2$ and $x^2 + y^2 = R^2$ respectively; and a sink has been substituted for the source at the coordinate origin. The equation of characteristics remains as before (5.2), and the field of characteristics corresponds to that shown on Fig. 4.

For the stress along characteristic $x = 0$ we have the following expression, different from (5.4):

$$\sigma_\xi = \left[\frac{\mu q}{k\varkappa}\frac{1}{y} - (1-\varepsilon)(\rho'-\rho)g\right](y - r_0) \qquad \text{for } x = 0 \tag{5.6}$$

It will be seen from (5.6) that with increasing flow rate q the change of the σ_ξ-stress sign will first occur at $y = r_0$. The corresponding minimum flow rate q_* is obviously determined by the condition that $(\partial\sigma_\xi/\partial y)_{y=r_0} = 0$, i.e. we have as before (5.5).

In this case the transition into the fluidized state is by way of a gradual 'erosion' of the upper surface of the bed, commencing at the central point of the free surface.

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